UDC 532.546

ON MOTION OF THE SEPARATION BOUNDARY OF FLUIDS IN NONLINEAR FILTRATION

PMM Vol.41, №4, 1977, pp. 754-758

O. V. GOLUBEVA and V. F. PIVEN'
(Moscow, Kostroma)

(Received December 22, 1975)

A forced piston-like displacement of one fluid by another under an arbitrary nonlinear filtration law, is considered. An exact solution of the problem is given for the one-dimensional case, and two approximate solutions for the two-dimensional case.

1. Let the medium be inhomogeneous and nondeformable, the fluid incompressible and its motion described by an arbitrary, in general nonlinear filtration law. Let the domain of filtration be bounded by two fixed boundaries L_s and L_k, and occupied by two fluids of different physical properties, displaced by a piston-like motion. The moving boundary L separating the fluids divides the filtration domain into two zones the filtration equations in which [1]

$$\operatorname{grad}\varphi_\alpha = F_\alpha \frac{\mathbf{v}_\alpha}{v_\alpha}, \quad \operatorname{div}\mathbf{v}_\alpha = 0 \tag{1.1}$$

$$\varphi_\alpha = -P_\alpha - \gamma_\alpha z, \quad \alpha = 1, 2$$

Here F_α is an experimental function depending on the rate of filtration modulus v_α, viscosity μ_α and density ρ_α of the fluids, permeability k and porosity m of the medium; P_α is the pressure, γ_α is the specific weight of the fluids and z is the vertical coordinate.

In the case of two-dimensional filtration in a layer of variable thickness $\sqrt{H}$ lying on the surface on which an orthogonal p, q-coordinate system is defined, Eqs. (1.1) become (see [2])

$$v_{\alpha p} = \frac{v_\alpha}{F_\alpha\sqrt{E}}\frac{\partial\varphi_\alpha}{\partial p} = \frac{1}{\sqrt{G}\sqrt{H}}\frac{\partial\psi_\alpha}{\partial q} \tag{1.2}$$

$$v_{\alpha q} = \frac{v_\alpha}{F_\alpha\sqrt{G}}\frac{\partial\varphi_\alpha}{\partial q} = -\frac{1}{\sqrt{E}\sqrt{H}}\frac{\partial\psi_\alpha}{\partial p}$$

$$v_\alpha = F_\alpha^{-1} = \frac{1}{\sqrt{H}}\sqrt{\frac{1}{G}\left(\frac{\partial\psi_\alpha}{\partial q}\right)^2 + \frac{1}{E}\left(\frac{\partial\psi_\alpha}{\partial p}\right)^2}$$

$$F_\alpha^{-1} = F_\alpha^{-1}(|\operatorname{grad}\varphi_\alpha|, \ \mu_\alpha, \rho_\alpha, k, m), \quad |\operatorname{grad}\varphi_\alpha| = \sqrt{\frac{1}{E}\left(\frac{\partial\varphi_\alpha}{\partial p}\right)^2 + \frac{1}{G}\left(\frac{\partial\varphi_\alpha}{\partial q}\right)^2}$$

F_α^{-1} is the inverse of F_α, $\sqrt{E}$ and $\sqrt{G}$ are the coefficients of the surface coordinate network and ψ_α is the stream function. The system (1.2) yields the following equations for φ_α:

$$\frac{\partial}{\partial p}\left(\frac{F_\alpha^{-1}\sqrt{G}\sqrt{H}}{|\operatorname{grad}\varphi_\alpha|\sqrt{E}}\frac{\partial\varphi_\alpha}{\partial p}\right) + \frac{\partial}{\partial q}\left(\frac{F_\alpha^{-1}\sqrt{E}\sqrt{H}}{|\operatorname{grad}\varphi_\alpha|\sqrt{G}}\frac{\partial\varphi_\alpha}{\partial q}\right) = 0 \tag{1.3}$$

If the boundaries L_s and L_k denote the contour of the hole and the feed, respectively, and constant pressures P_α are specified at these contours, then the conditions for φ_α will be

$$[\varphi_1]_{L_s} = \varphi_s, \quad [\varphi_2]_{L_k} = \varphi_k \tag{1.4}$$

The conditions of continuity of pressure and normal velocity components which hold at the boundary L, can be written in the form

$$[\varphi_1 + \gamma_1 z]_L = [\varphi_2 + \gamma_2 z]_L \tag{1.5}$$

$$\left[\frac{F_1^{-1}}{|\operatorname{grad}\varphi_1|}\frac{\partial\varphi_1}{\partial n}\right]_L = \left[\frac{F_2^{-1}}{|\operatorname{grad}\varphi_2|}\frac{\partial\varphi_2}{\partial n}\right]_L$$

We shall seek the equation of the boundary L in the parametric form

$$p_L = p_L(t, \tau_0), \quad q_L = q_L(t, \tau_0)$$

where τ_0 is the parameter. At the initial instant $t = 0$ the equation of the boundary is known, and is

$$p_0 = p_L(0, \tau_0), \quad q_0 = q_L(0, \tau_0) \tag{1.6}$$

The physical velocity of the fluid particles $d\mathbf{r} / dt$ and the rate of filtration $\mathbf{v}$ are connected by the relation $d\mathbf{r} / dt = \mathbf{v} / m$, therefore the differential equations of the boundary L will have the form

$$\frac{dp_L}{dt} = \left[\frac{F_\alpha^{-1}\partial\varphi_\alpha/\partial p}{|\operatorname{grad}\varphi_\alpha|\, m\sqrt{E}}\right]_{p=p_L,\ q=q_L} \tag{1.7}$$

$$\frac{dq_L}{dt} = \left[\frac{F_\alpha^{-1}\partial\varphi_\alpha/\partial q}{|\operatorname{grad}\varphi_\alpha|\, m\sqrt{G}}\right]_{p=p_L,\ q=q_L}, \quad \alpha = 1 \text{ or } 2$$

The above equations should be integrated together with: (1.3), the boundary conditions (1.4), (1.5) and the initial condition (1.6), this is a difficult problem.

2. In the particular case of one-dimensional filtration the problem posed above admits an exact solution in a finite form. Namely, the filtration will take place along the lines $p =$ const, provided that the following restrictions are imposed on the surface coordinate network and on the laws of variation in the thickness permeability and porosity of the layer:

$$\sqrt{E}\sqrt{H} = A(p)B(q), \quad \sqrt{G} = C(q) \tag{2.1}$$
$$z = z(q), \quad k = k(q), \quad m = m(q)$$

Here $A(p)$ and $B(q)$, $C(q)$ are some functions of p and q. The contours of the hole L_s and the feed L_k will be situated along the lines $q = q_s$ and $q = q_k$, and the boundary L will follow the lines $q =$ const for the initial and subsequent instants of time.

In this case Eqs. (1.3) and conditions (1.4), (1.5) together yield

$$\varphi_1 = \varphi_s + \int_{q_s}^{q} F_1 dq, \quad \varphi_2 = \varphi_k + \int_{q_k}^{q} F_2 dq \tag{2.2}$$

from which it follows that the flow rate will be

$$v_{1q} = v_{2q} = v = \lambda(q_L) A(p) / (\sqrt{E}\sqrt{H})$$

where $\lambda\ (q_L)$ is determined by the equation

$$\int_{q_s}^{q_L} \sqrt{G}F_1 dq + \int_{q_L}^{q_k} \sqrt{G}F_2 dq + \varphi_s - \varphi_k + (\gamma_1 - \gamma_2) z = 0$$

$$F_\alpha = F_\alpha (\lambda\ (q_L) A\ (p)\ /\ (\sqrt{E}\ \sqrt{H}),\ \mu_\alpha,\ \rho_\alpha,\ k,\ m), \qquad \alpha = 1,\ 2$$

Then from (1.6) and (1.7) it follows that the determination of the boundary L separating the fluids can be reduced to the problem of computing the quadrature

$$t = \frac{1}{A\ (p)} \int_{q_0}^{q_L} [m\sqrt{EGH}\,]_{q=q_L} \frac{dq_L}{\lambda\ (q_L)} \tag{2.3}$$

3. In the case of two-dimentional filtration two consecutive approximate solutions in a finite form can be obtained for the problem of displacement of the boundary separating the fluids.

In the first approximation we assume that the physical properties of the fluids are the same (one fluid system) and, that a solution of (1.3) can be found in the form

$$\varphi = \Phi\ (\Lambda,\ p,\ q) + M \tag{3.1}$$

where Λ and M are constants obtained from the boundary condition (1.4). Then the system (1.2) will yield the stream function $\psi\ (p,\ q)$.

Since in the present case the boundary L consists of the specified fluid particles, the equation of streamlines $\psi\ (p,\ q) = \psi\ (p_0,\ q_0) = a$, or in the parametric form

$$p = p\ (\tau,\ a), \qquad q = q\ (\tau,\ a) \qquad (a = a\ (\tau_0) = \text{const}) \tag{3.2}$$

is the first integral of the differential equations (1.7). Then the equation of the boundary L becomes

$$p_L = p\ (\tau_L,\ a), \qquad q_L = q\ (\tau_L,\ a) \tag{3.3}$$

where the parameter τ_L defining the position of the point of the boundary on the streamline (3.2) should be determined as a function of time. In accordance with (1.7), the equation of motion of a point of the boundary along the streamline (3.2) is

$$[dS\ /\ dt]_{\tau=\tau_L} = [F^{-1}\ /\ m]_{\tau=\tau_L}$$

or

$$\frac{d\tau_L}{dt} = \left[\frac{F^{-1}}{m dS\ /d\tau}\right]_{\tau=\tau_L} \qquad \left(\frac{dS}{dt} = \frac{dS}{d\tau}\frac{d\tau}{dt}\right) \tag{3.4}$$

$$F^{-1} = F^{-1}\left(\frac{\partial\varphi}{\partial\tau}\frac{1}{dS/d\tau},\ \mu,\ \rho,\ k,\ m\right), \quad \frac{dS}{d\tau} = \sqrt{E\left(\frac{dp}{d\tau}\right)^2 + G\left(\frac{dq}{d\tau}\right)^2}$$

where k and m can be written, with the help of (3.2), as functions of τ and a. We shall count τ_L from the initial position of the boundary L. Then the equation (3.4) will have to be integrated with the initial condition $\tau_L\,|_{t=0} = 0$, and this gives

$$t = \int_0^{\tau_L} \left[\frac{m}{F^{-1}}\frac{dS}{d\tau}\right]_{\tau=\tau_L} d\tau_L \tag{3.5}$$

from which we have

$$\tau_L = \tau_L^{(1)}(t, a) \tag{3.6}$$

Then (3.3) and (3.6) together will yield the equation of the boundary L in the first approximation. In the second approximation we adopt the scheme of rigid streamlines [3]. The stream function of the first approximation is used here again. Since the streamlines do not refract in the boundary L, the second condition of (1.5) of continuity of the normal velocity components can be replaced by the condition that the velocity vectors are equal to each other, i.e.

$$\left[\frac{F_1^{-1}\operatorname{grad}\varphi_1}{|\operatorname{grad}\varphi_1|}\right]_L = \left[\frac{F_2^{-1}\operatorname{grad}\varphi_2}{|\operatorname{grad}\varphi_2|}\right]_L \tag{3.7}$$

Consequently the first condition of (1.5) and (3.7) will hold at the boundary L. We shall write this condition in the form of a projection on the tangent to the streamline

$$[\varphi_1 + \gamma_1 z]_{\tau=\tau_L} = [\varphi_2 + \gamma_2 z]_{\tau=\tau_L}, \quad [F_1^{-1}]_{\tau=\tau_L} = [F_2^{-1}]_{\tau=\tau_L} \tag{3.8}$$

Making use of the solution (3.1), we seek the function φ_α along the streamline (3.2) in form

$$\varphi_\alpha = \Phi_\alpha(\Lambda_\alpha(\tau_L), p, q) + M_\alpha(\tau_L), \quad \alpha = 1, 2 \tag{3.9}$$

where Λ_α and M_α are determined as functions of τ_L from the boundary conditions (1.4) and (3.8). Substituting (3.9) into (3.5), we obtain

$$\tau_L = \tau_L^{(2)}(t, a) \tag{3.10}$$

Then (3.3) and (3.10) will together yield the equation of the boundary L in the finite form, in the second approximation.

4. As an example we shall consider the motion of the boundary separating the fluids towards a real operational well, in a plane ($\sqrt{E} = r$, $\sqrt{G} = 1$) homogeneous stratum ($k = \text{const}$, $m = \text{const}$). The following conditions are given at the well of radius r_s and at the concentric feed contour of radius r_k:

$$[\varphi_1]_{r=r_s} = \varphi_s, \quad [\varphi_2]_{r=r_k} = \varphi_k \tag{4.1}$$

The equation of the boundary at the initial instant of time is known

$$r_0 = r_0(\tau_0), \quad \theta_0 = \theta_0(\tau_0) \tag{4.2}$$

Let us solve the problem in the first approximation. The flow is uniform, therefore (1.3) yields the following function satisfying the conditions (1.4):

$$\varphi = \varphi_s - \int_{r_s}^{r} F\left(\frac{Q}{2\pi r}\right) dr \quad \left(\int_{r_s}^{r_k} F\left(\frac{Q}{2\pi r}\right) dr - (\varphi_s - \varphi_k) = 0\right) \tag{4.3}$$

Here Q denotes the output of the well per unit thickness of the stratum, and can be found from the relation given in brackets. In this case it is expedient to choose the parameter τ_L in the form $\tau_L = r_0 - r_L$ where r_L is the distance between the boundary L and the center of the well. Then in accordance with (3.5), (4.2) and (4.3), we have the following equation of the boundary in the first approximation:

$$r_L = \sqrt{r_0^2(\tau_0) - \frac{Qt}{\pi m}}, \quad \theta_0 = \theta_0(\tau_0) \tag{4.4}$$

Next we obtain the solution of the problem in the second approximation. At the boundary L the conditions (3.8) assume the form

$$[\varphi_1]_{r=r_L} = [\varphi_2]_{r=r_L}, \quad [F_1^{-1}]_{r=r_L} = [F_2^{-1}]_{r=r_L} \tag{4.5}$$

In this case the function φ_α satisfying the conditions (4.1) and (4.5), becomes

$$\varphi_1 = \varphi_s - \int_{r_s}^{r} F_1 dr, \quad \varphi_2 = \varphi'_k + \int_{r}^{r_k} F_2 dr \tag{4.6}$$

$$F_\alpha = F_\alpha\left(\frac{Q(r_L)}{2\pi r}, \mu_\alpha, \rho_\alpha\right) \quad \left(\int_{r_s}^{r_L} F_1 dr + \int_{r_L}^{r_k} F_2 dr - (\varphi_s - \varphi_k) = 0\right)$$

and the equation of the boundary L in the second approximation is

$$t = 2\pi m \int_{r_L}^{r_0(\tau_0)} \frac{r_L dr_L}{Q(r_L)}, \quad \theta_0 = \theta_0(\tau_0) \tag{4.7}$$

5. We use the results obtained to investigate the effect of the nonlinearity of the filtration law on the motion of the boundary L.

We shall only deal with the first approximation. Let the filtration be governed by the following two-term law [4]:

$$F = \frac{\mu}{k} v + \beta v^2 \tag{5.1}$$

and let the boundary L at the initial instant be a straight line separated from the ox-axis by the distance h

$$r_0 \sin\theta_0 = h \quad (0 \leqslant \theta_0 \leqslant \pi)$$

According to (4.4) the equation of the boundary is

$$r_L = \sqrt{\frac{h^2}{\sin^2\theta_0} - \frac{Qt}{\pi m}}, \quad Q = 4\pi(\varphi_s - \varphi_k)\left(\frac{\mu}{k}\ln\frac{r_k}{r_s}\right)^{-1}(1 + \sqrt{1+b})^{-1}$$

$$b = 4\beta(\varphi_s - \varphi_k)\left(\frac{1}{r_s} - \frac{1}{r_k}\right)\left(\frac{\mu}{k}\ln\frac{r_k}{r_s}\right)^{-2}$$

The boundary will reach the well most rapidly by moving in the shortest direction $\theta_0 = \pi/2$. If we assume that $r_s \ll h$, the displacement of the boundary in this direction will be

$$r_L = h\sqrt{1 - \frac{2}{b}(\sqrt{1+b} - 1)\frac{t}{T}}, \quad T = \frac{m\mu h^2}{2k(\varphi_s - \varphi_k)}\ln\frac{r_k}{r_s}$$

where T denotes the time in which the boundary will reach the well when the filtration is linear.

Figure 1 shows the displacement of the boundary in the direction $\theta_0 = \pi/2$ when the filtration is linear, i.e. when $\beta = 0$ and consequently $b = 0$, and for the nonlinear filtration when $b = 1, 5$ and 10. We see that the nonlinearity of the filtration law (5.1) slows down the motion of the boundary.

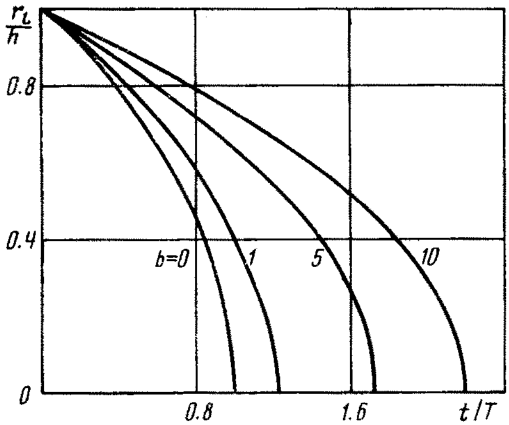

Fig. 1

REFERENCES

1. Khristianovich S. A., Motion of ground water not obeying Darcy's Law. PMM Vol. 4, № 1, 1940.
2. Golubeva O. V., Equations of two-dimensional motions of an ideal fluid along a curved surface and their application to the theory of filtration. PMM Vol. 14, № 3, 1950.
3. Charnyi I. A., Subterranean Hydro-gas-dynamics. Moscow, Gosoptekhizdat, 1963.
4. Progress of the Studies in the Theory of Filtration in SSSR. (1917 - 1967). Moscow, "Nauka", 1969.

Translated by L. K.